# Quaternions

- William Rowan Hamilton (1805-1865)
  - Algebraic couples (complex number) 1833

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  - Quaternions 1843

$$w + ix + jy + kz \quad \textbf{where} \quad \begin{gathered} i^2 = j^2 = k^2 = ijk = -1 \\ ij = k, \quad jk = i, \quad ki = j \\ ji = -k, kj = -i, ik = -j \end{gathered}$$

# Quaternions

## William Thomson

***"… though beautifully ingenious, have been an unmixed evil to those who have touched them in any way."***

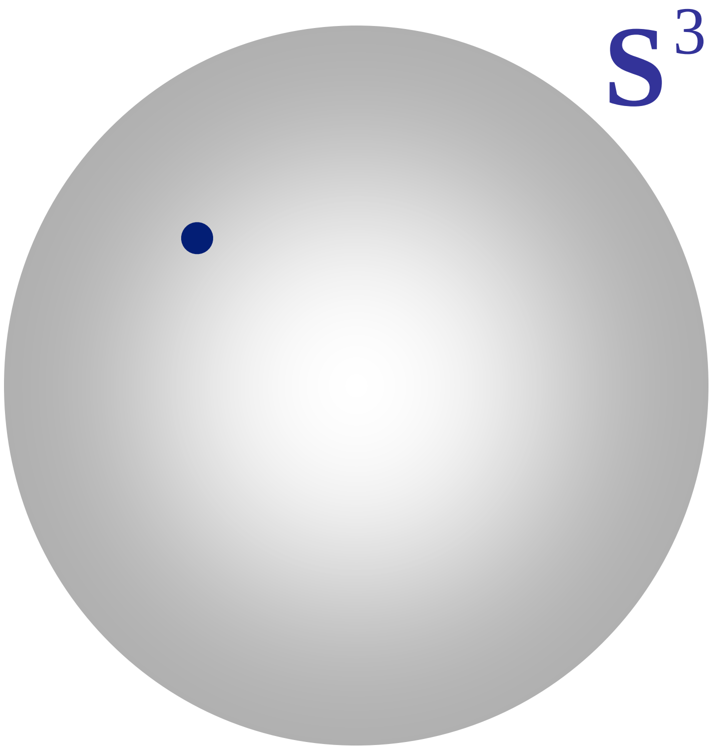
## Arthur Cayley

***"… which contained everything but had to be unfolded into another form before it could be understood."***

# Unit Quaternions

- Unit quaternions represent 3D rotations

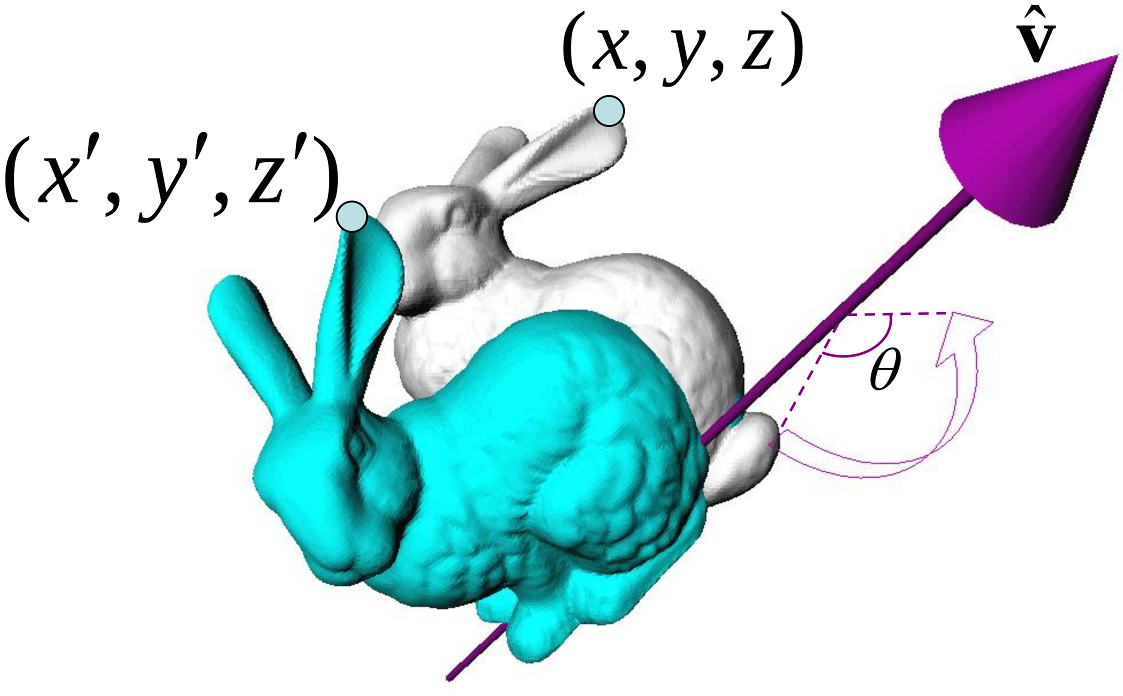
$$
\begin{aligned}
\mathbf{q} &= w + ix + jy + kz \\
&= (w, x, y, z) \\
&= (w, \mathbf{v})
\end{aligned}
$$




$$w^2 + x^2 + y^2 + z^2 = 1$$

# Rotation about an Arbitrary Axis

- Rotation about axis $\hat{\mathbf{v}}$ by angle $\theta$




$$\mathbf{q} = \left(\cos\frac{\theta}{2}, \hat{\mathbf{v}}\sin\frac{\theta}{2}\right)$$

$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$ **where** $\mathbf{p} = (0, x, y, z)$

Purely Imaginary Quaternion

# Unit Quaternion Algebra

- Identity

$$\mathbf{q} = (1,0,0,0)$$

- Multiplication

$$\mathbf{q}_1\mathbf{q}_2 = (w_1, \mathbf{v}_1)(w_2, \mathbf{v}_2)$$
$$= (w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, w_1\mathbf{v}_2 + w_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)$$

- Inverse

$$\mathbf{q}^{-1} = (w,-x,-y,-z)/(w^2 + x^2 + y^2 + z^2)$$
$$= (-w, x, y, z)/(w^2 + x^2 + y^2 + z^2)$$

- Unit quaternion space is
  - closed under multiplication and inverse,
  - but not closed under addition and subtraction

# Unit Quaternion Algebra

- Antipodal equivalence
  - q and –q represent the same rotation

$$R_{\mathbf{q}}(\mathbf{p}) = R_{-\mathbf{q}}(\mathbf{p})$$

  - 2-to-1 mapping between $\mathbf{S^3}$ and **SO(3)**
  - Twice as fast as in **SO(3)**

# Rotation Composition

- Rotation by a matrix

$$v' = Mv$$

- Rotation by a unit quaternion

$$v' = qvq^{-1}$$

- Composition of Matrices (or Unit quaternions) is simple multiplication

$$v' = M_2 M_1 v$$

$$v' = q_2 q_1 v q_1^{-1} q_2^{-1}$$