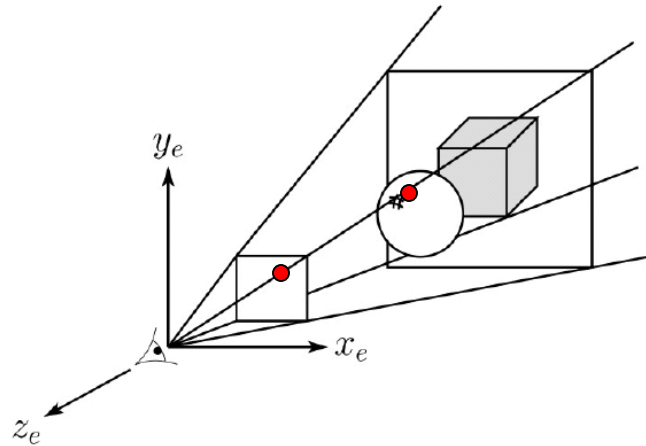


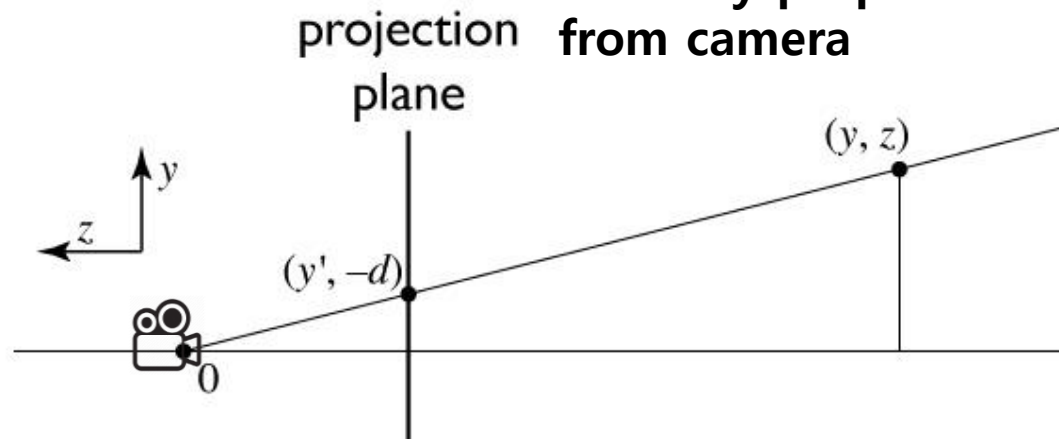
Let's first consider 3D View Frustum \rightarrow 2D Projection Plane

- Consider the projection of a 3D point on the camera plane



Perspective projection

The size of an object on the screen is inversely proportional to its distance from camera



similar triangles:

$$\frac{y'}{d} = \frac{y}{-z}$$

$$y' = -dy/z$$

Homogeneous coordinates revisited

- Perspective requires division
 - that is not part of affine transformations
 - in affine, parallel lines stay parallel
 - therefore not vanishing point
 - therefore no rays converging on viewpoint
- “True” purpose of homogeneous coords: projection

Homogeneous coordinates revisited

- Introduced $w = 1$ coordinate as a placeholder

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

– used as a convenience for unifying translation with linear

- Can also allow arbitrary w

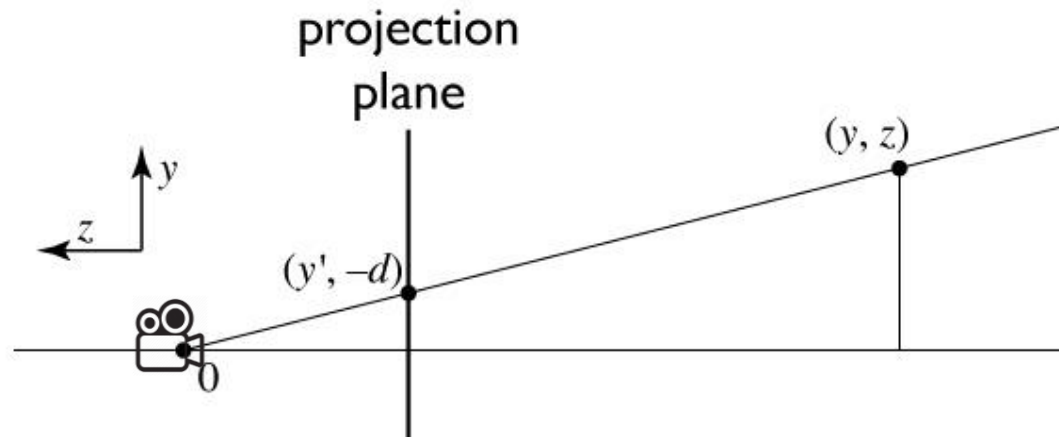
$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \sim \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}$$

Implications of w

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \sim \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}$$

- All scalar multiples of a 4-vector are equivalent
- When w is not zero, can divide by w
 - therefore these points represent “normal” affine points
- When w is zero, it’s a point at infinity, a.k.a. a direction
 - this is the point where parallel lines intersect
 - can also think of it as the vanishing point
- Digression on projective space

Perspective projection

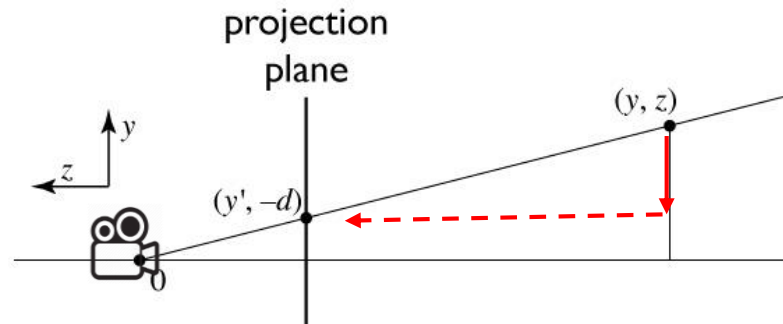


to implement perspective, just move z to w :

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -dx/z \\ -dy/z \\ 1 \end{bmatrix} \sim \begin{bmatrix} dx \\ dy \\ -z \end{bmatrix} = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

So far, 3D View Frustum \rightarrow 2D Projection Plane

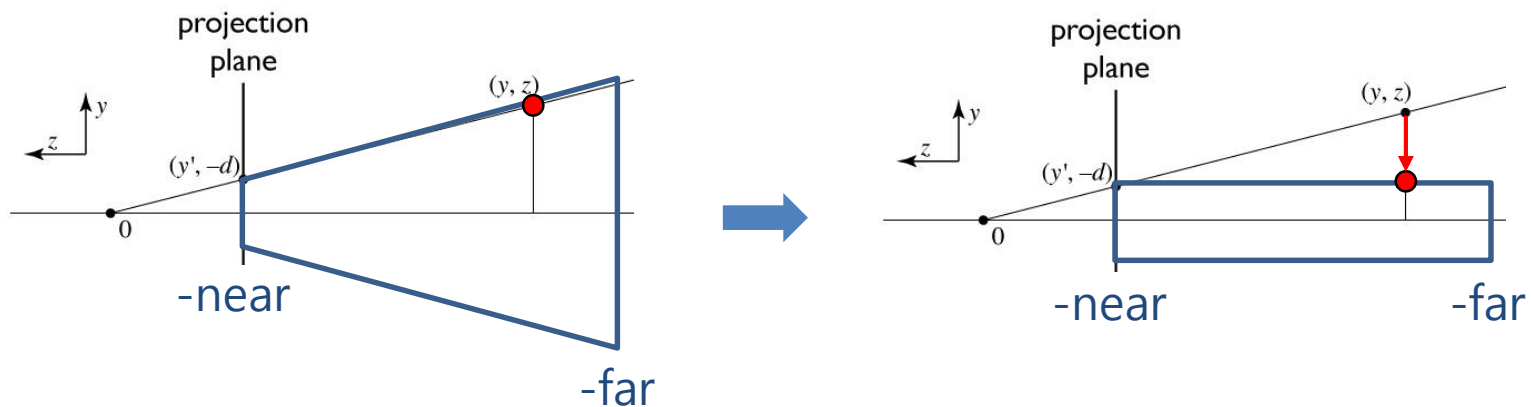
- What we've just seen is a story of 3D \rightarrow 2D



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -dx/z \\ -dy/z \\ 1 \end{bmatrix} \sim \begin{bmatrix} dx \\ dy \\ -z \end{bmatrix} = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Now, 3D View Frustum \rightarrow 3D Cuboid

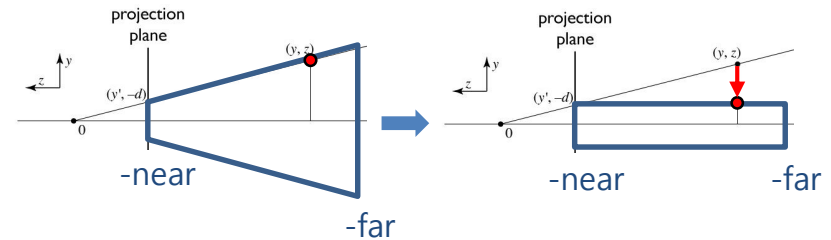
- What we have to do is 3D \rightarrow 3D
 - Let's first consider a viewing frustum \rightarrow a cuboid with the same near and far offset (not a canonical view volume)



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -dx/z \\ -dy/z \\ -X/z \\ 1 \end{bmatrix} \sim \begin{bmatrix} dx \\ dy \\ X \\ -z \end{bmatrix} = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ c_0 & c_1 & a & b \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D View Frustum \rightarrow 3D Cuboid

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -dx/z \\ -dy/z \\ -X/z \\ 1 \end{bmatrix} \sim \begin{bmatrix} dx \\ dy \\ X \\ -z \end{bmatrix} = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ c_0 & c_1 & a & b \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



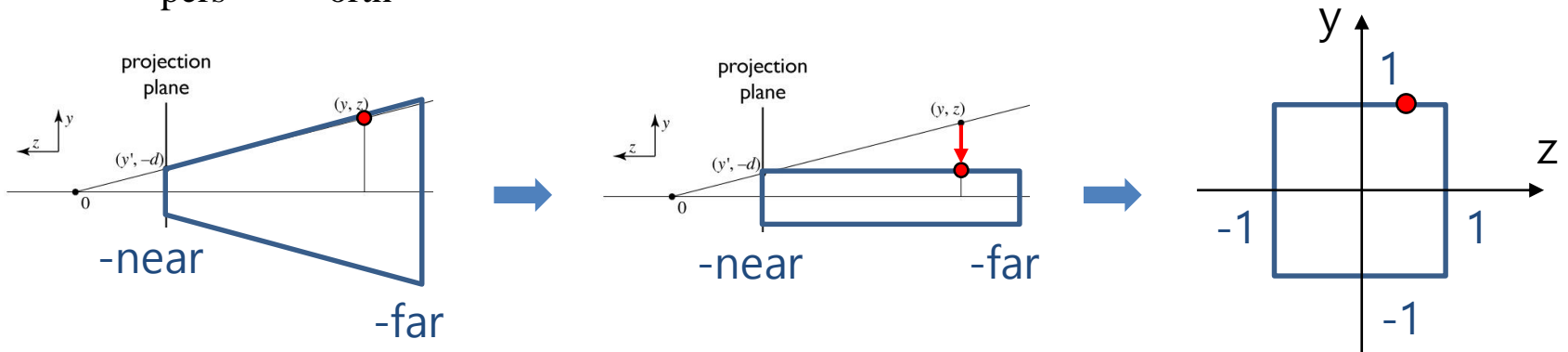
- Note that z' is independent of x and y , therefore $c_0 = c_1 = 0$
- We want z depth $-near \rightarrow -near$, $-far \rightarrow -far$

$$X = az + b \quad z'(z) = -X/z = -\frac{az + b}{z}$$
- Find 2 unknowns a , b with 2 eq. $z'(-n)=-n$, $z'(-f)=-f$
- $\rightarrow a = f+n$, $b = fn$ (try it)

Final: 3D View Frustum → 3D Canonical View Volume

- By substituting d with n , $P = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & f + n & fn \\ 0 & 0 & -1 & 0 \end{bmatrix}$
- Now the remaining work is mapping the cuboid to a canonical view volume: M_{orth}

- Viewing frustum → cuboid → canonical view volume:
 $M_{\text{pers}} = M_{\text{orth}} P$



Perspective Projection Matrix

- $M_{\text{pers}} = M_{\text{orth}} P$

$$= \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{-2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & f+n & fn \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Acknowledgement

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 - Prof. Taesoo Kwon, Hanyang Univ., <http://calab.hanyang.ac.kr/cgi-bin/cg.cgi>
 - Prof. Steve Marschner, Cornell Univ., <http://www.cs.cornell.edu/courses/cs4620/2014fa/index.shtml>