Quaternions

- William Rowan Hamilton (1805-1865)
  - Algebraic couples (complex number) 1833

$x + iy$ **where** $i^2 = -1$

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  - Quaternions 1843

$$w + ix + jy + kz \quad \textbf{where} \quad \begin{gathered} i^2 = j^2 = k^2 = ijk = -1 \\ ij = k, \quad jk = i, \quad ki = j \\ ji = -k, kj = -i, ik = -j \end{gathered}$$

Quaternions

William Thomson

"… though beautifully ingenious, have been an unmixed evil to those who have touched them in any way."

Arthur Cayley

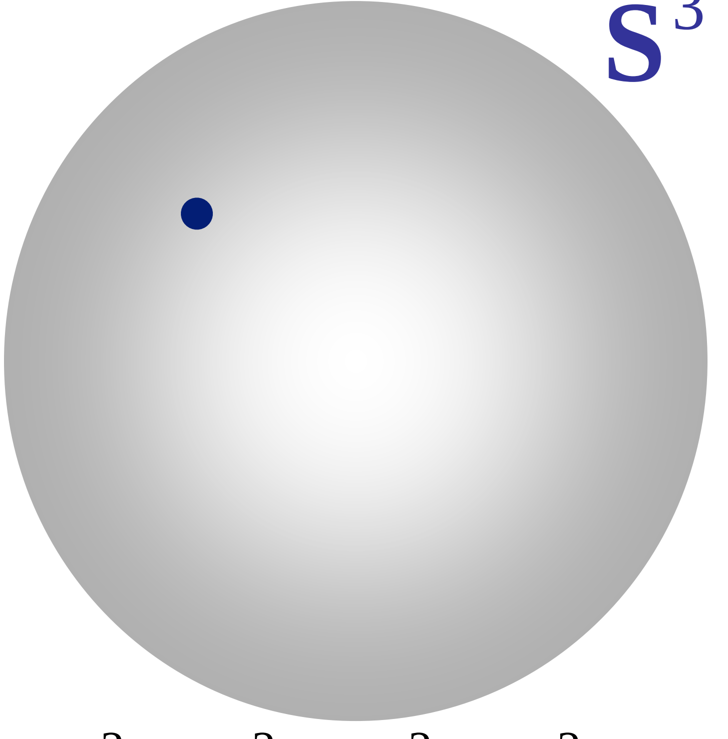
"… which contained everything but had to be unfolded into another form before it could be understood."

Unit Quaternions

- Unit quaternions represent 3D rotations

$$
\begin{aligned}
\mathbf{q} &= w + ix + jy + kz \\
&= (w, x, y, z) \\
&= (w, \mathbf{v})
\end{aligned}
$$

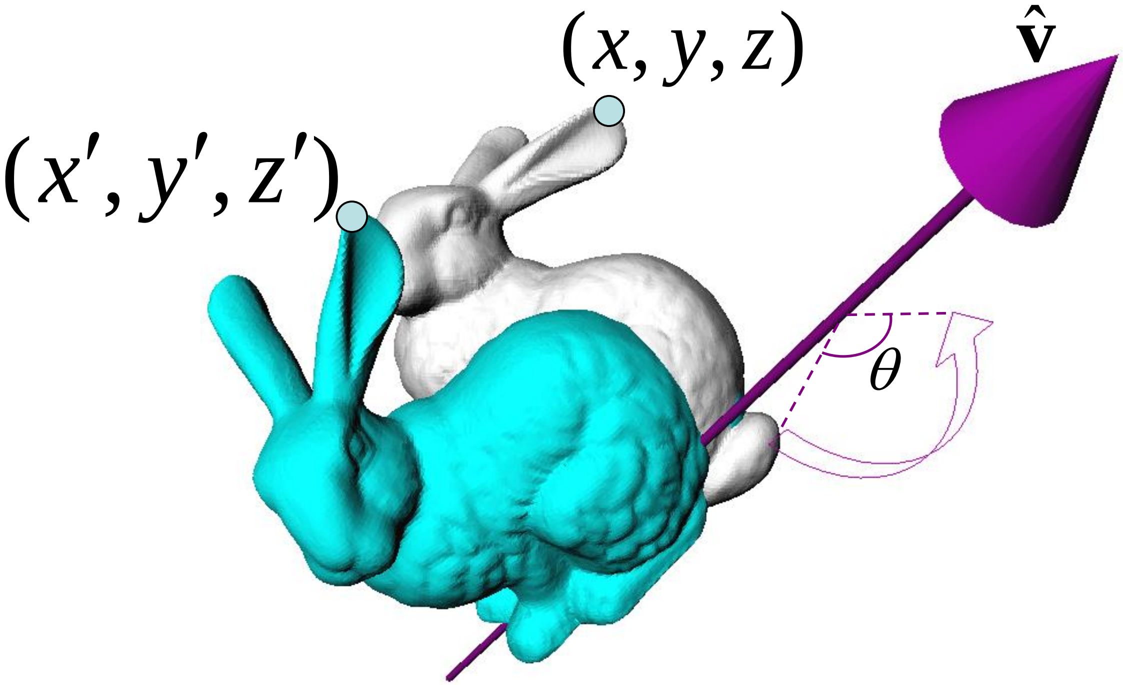
$\mathbf{S}^3$



$$w^2 + x^2 + y^2 + z^2 = 1$$

Rotation about an Arbitrary Axis

- Rotation about axis $\hat{\mathbf{v}}$ by angle θ



$$\mathbf{q} = \left(\cos\frac{\theta}{2}, \hat{\mathbf{v}} \sin\frac{\theta}{2} \right)$$

$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$ **where** $\mathbf{p} = (0, x, y, z)$

Purely Imaginary Quaternion

Unit Quaternion Algebra

- Identity

$$\mathbf{q} = (1,0,0,0)$$

- Multiplication

$$\begin{aligned}\mathbf{q}_1\mathbf{q}_2 &= (w_1, \mathbf{v}_1)(w_2, \mathbf{v}_2) \\ &= (w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, w_1\mathbf{v}_2 + w_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)\end{aligned}$$

- Inverse

$$\begin{aligned}\mathbf{q}^{-1} &= (w,-x,-y,-z)/(w^2 + x^2 + y^2 + z^2) \\ &= (-w, x, y, z)/(w^2 + x^2 + y^2 + z^2)\end{aligned}$$

- Unit quaternion space is
  - closed under multiplication and inverse,
  - but not closed under addition and subtraction

Unit Quaternion Algebra

- Antipodal equivalence
  - q and –q represent the same rotation

$$R_{\mathbf{q}}(\mathbf{p}) = R_{-\mathbf{q}}(\mathbf{p})$$

  - 2-to-1 mapping between $\mathbf{S^3}$ and **SO(3)**
  - Twice as fast as in **SO(3)**

Rotation Composition

- Rotation by a matrix

$$v' = Mv$$

- Rotation by a unit quaternion

$$v' = qvq^{-1}$$

- Composition of Matrices (or Unit quaternions) is simple multiplication

$$v' = M_2 M_1 v$$

$$v' = q_2 q_1 v q_1^{-1} q_2^{-1}$$

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