Computer Graphics

3 - Transformation 1

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Topics Covered

- 2D Transformation
  - Scale, rotation, translation…

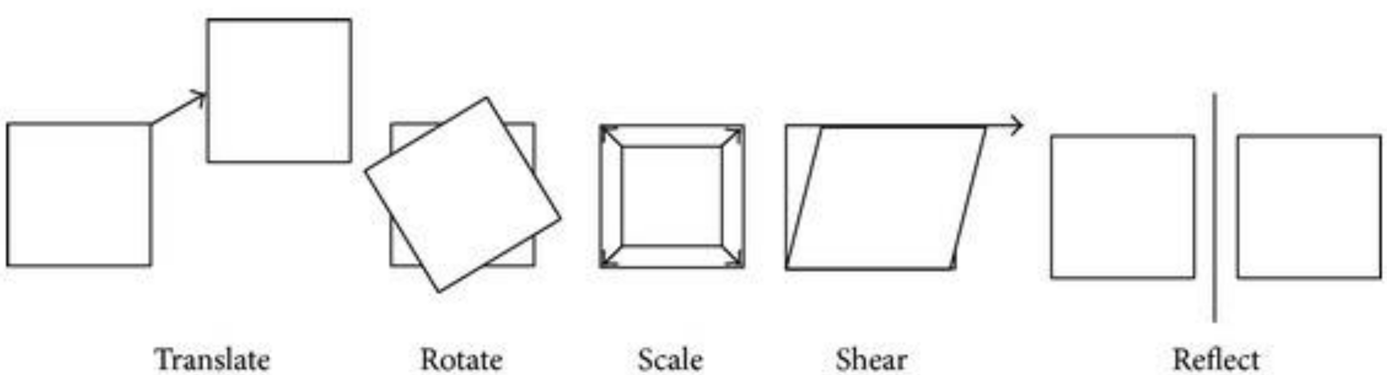
- Composing Transformations & Homogeneous Coordinates

- 3D Cartesian Coordinate System

2D Transformations

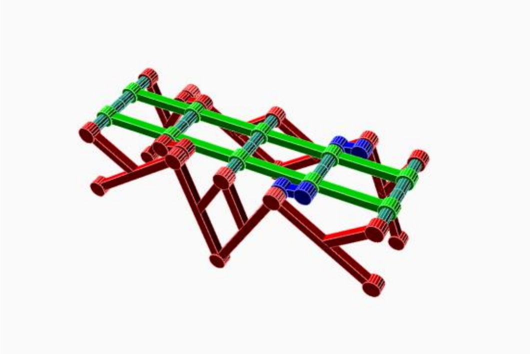
What is Transformation?

- Geometric **Transformation** - 기하 **변환**
  - One-to-one mapping (function) of a set having some geometric structure to itself or another such set.
  - More easily, *“moving a set of points”*
- Examples:



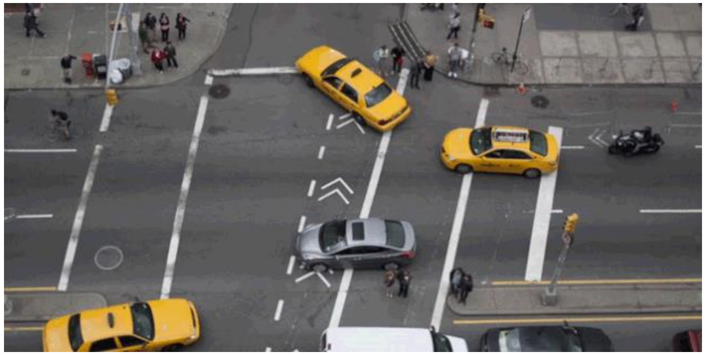
Where are Transformations used?

- Movement

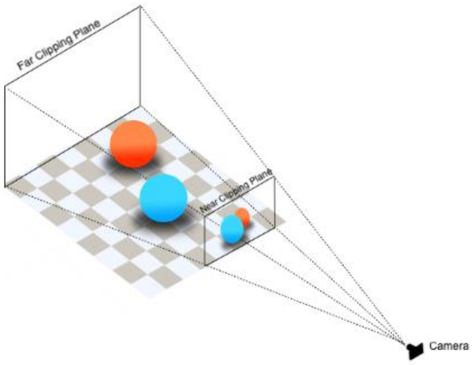


https://upload.wikimedia.org/wikipedia/commons/0/05/Extra_Simple_Walker_3D_Animation.gif

- Image/object manipulation



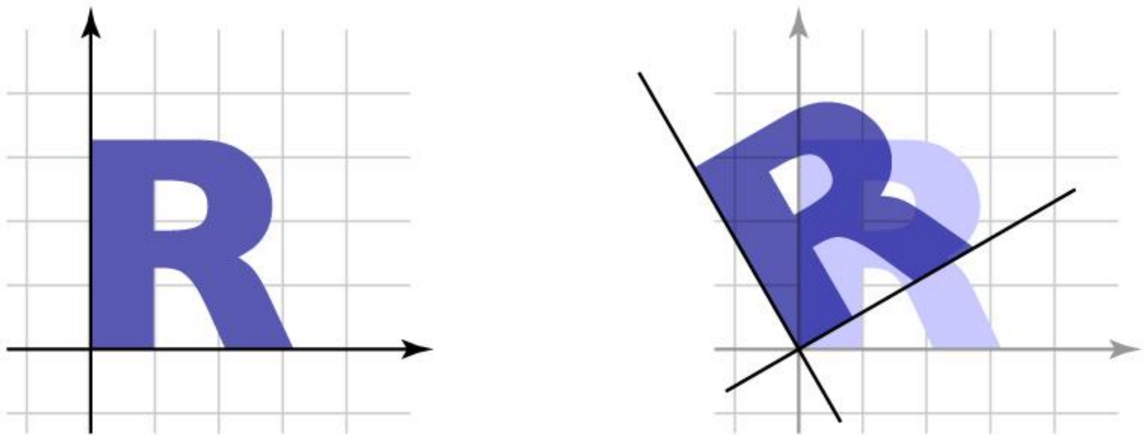
- Viewing, projection transform



Transformation

- "Moving a set of points"
  - Transformation T maps any input vector v in the vector space S to T(v).

$$S \rightarrow \{T(\mathbf{v}) \,|\, \mathbf{v} \in S\}$$



Linear Transformation

- One way to define a transformation is by matrix multiplication:

$$T(\mathbf{v}) = M\mathbf{v}$$

- This is called a **linear transformation** because a matrix multiplication represents a linear mapping.

$$T(a\mathbf{u} + \mathbf{v}) = aT(\mathbf{u}) + T(\mathbf{v})$$

$$\mathbf{M} \cdot (a\mathbf{u} + \mathbf{v}) = a\mathbf{M}\mathbf{u} + \mathbf{M}\mathbf{v}$$

2D Linear Transformation

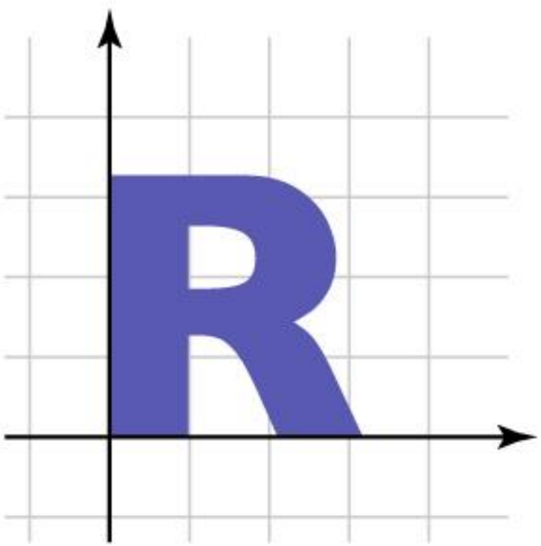
- 2x2 matrices represent 2D linear transformations such as:
  - uniform scale
  - non-uniform scale
  - rotation
  - shear
  - reflection

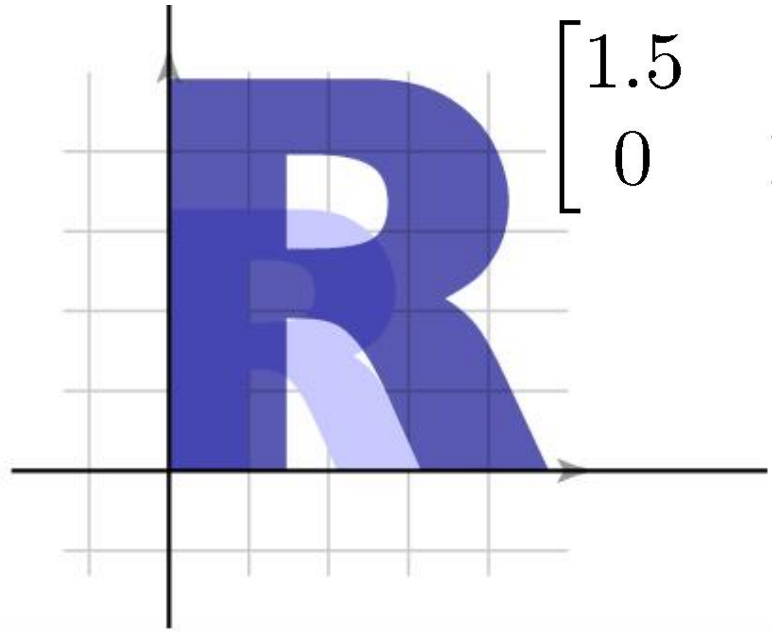
2D Linear Trans. – Uniform Scale

- Uniformly shrinks or enlarges both in x and y directions.

$$\underbrace{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{p} \underset{=}{=} \underbrace{\begin{bmatrix} sx \\ sy \end{bmatrix}}_{p'}$$

2x2 scale matrix



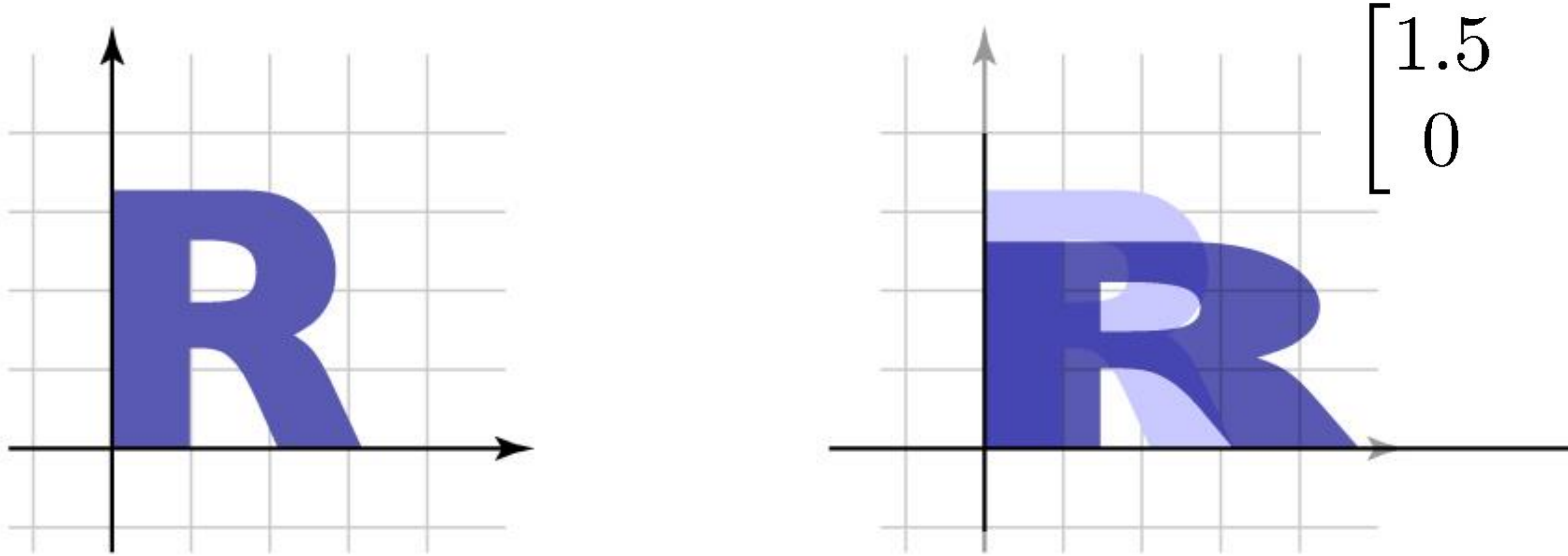


$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

2D Linear Trans. – Nonuniform Scale

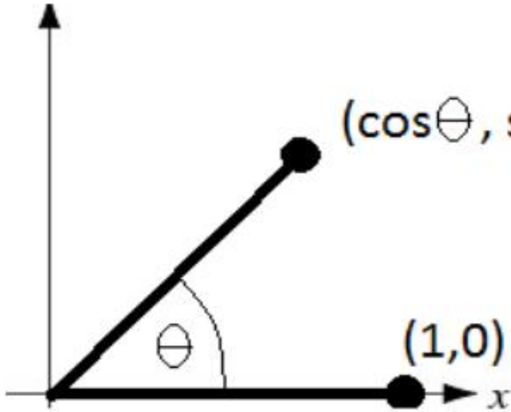
- Non-uniformly shrinks or enlarges in x and y directions.

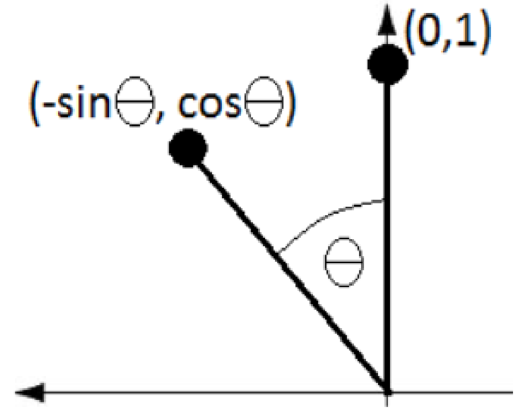
$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.8 \end{bmatrix}$$

Rotation



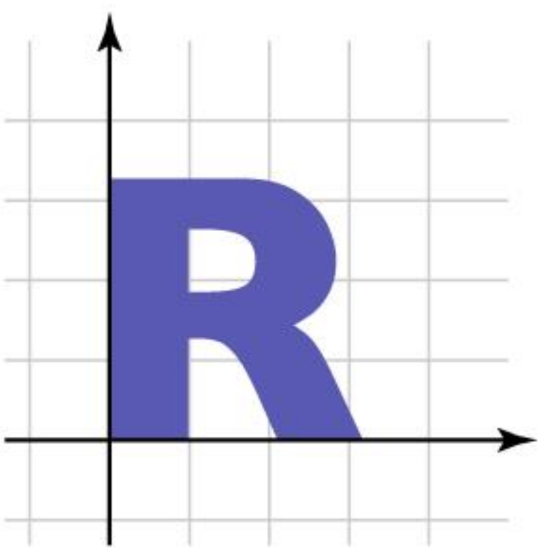


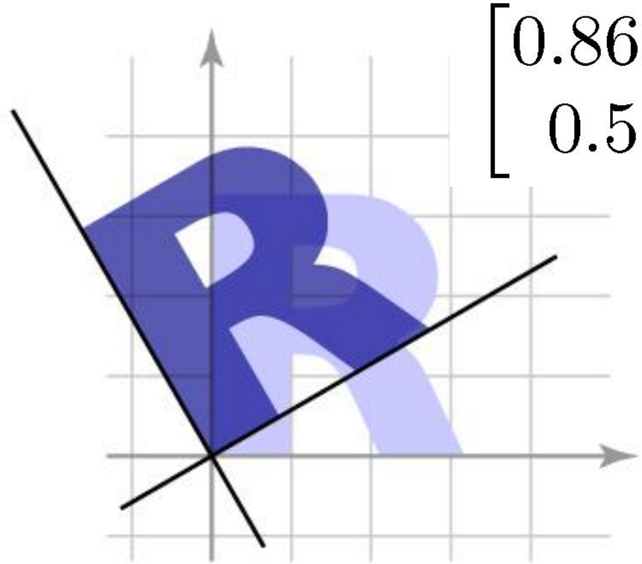
$$\Rightarrow R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$: Rotation matrix

2D Linear Trans. – Rotation

- Rotation can be written in matrix multiplication, so it's also a linear transformation.
  - Note that positive angle means CCW rotation.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$



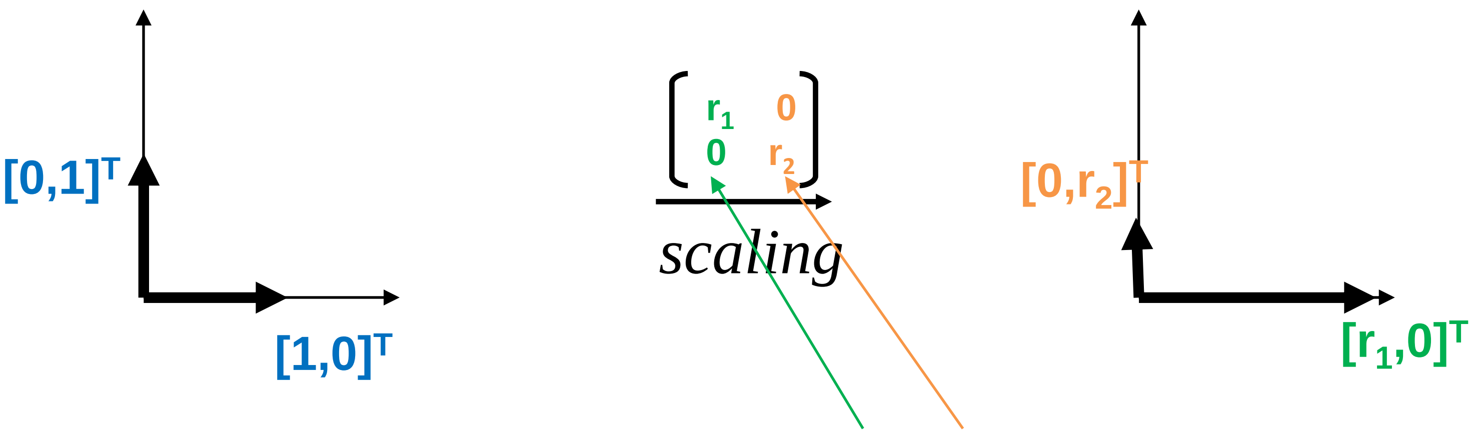


$$\begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}$$

(Rotate 30 deg ccw)

Numbers in Matrices: Scale, Rotation

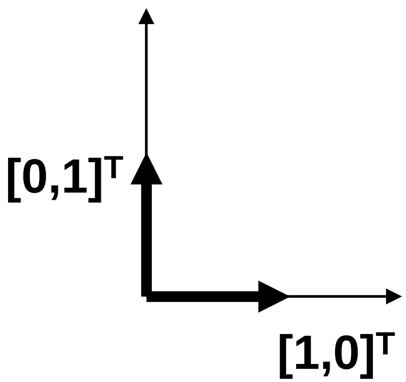
- Let's think about what the numbers in the matrix means.



Canonical basis vectors: unit vectors pointing in the direction of the axes of a Cartesian coordinate system.

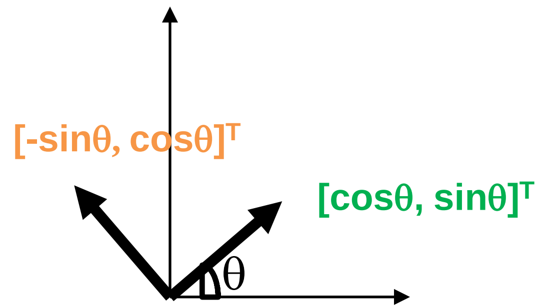
1st & 2nd basis vector of the transformed coordinates

Numbers in Matrices: Scale, Rotation



$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

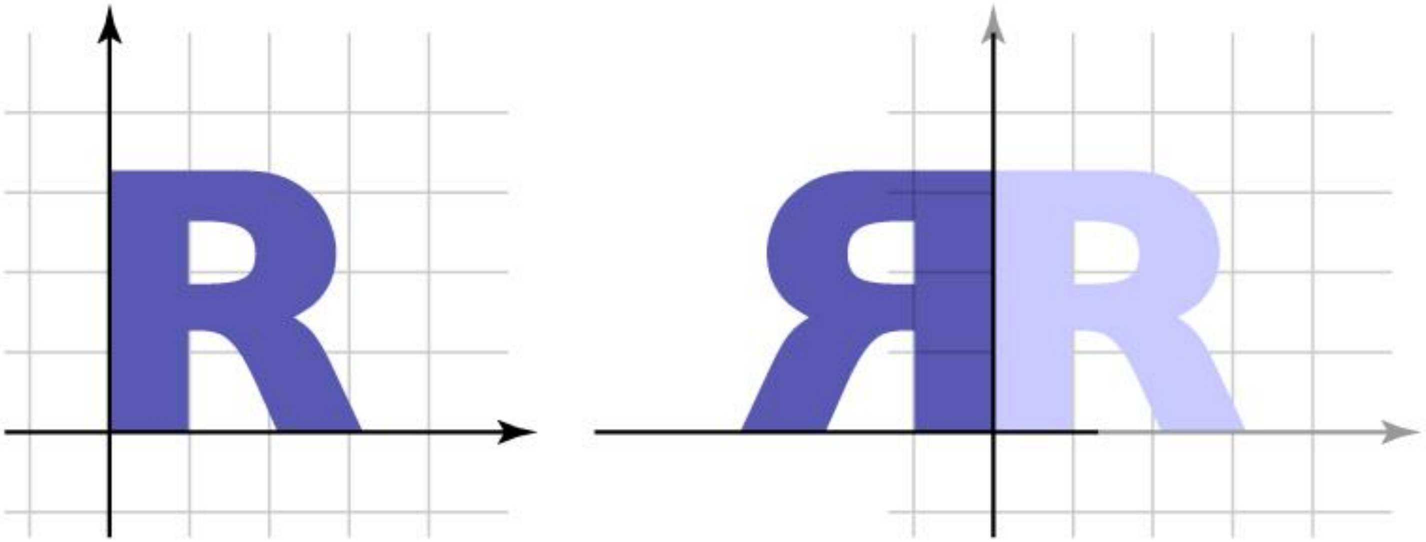
rotation



- *Column vectors* of a matrix is the *basis vectors of the column space (range)* of the matrix.
  - *Column space* of a matrix: The span (a set of all possible linear combinations) of its column vectors.

2D Linear Trans. – Reflection

- Reflection can be considered as a special case of non-uniform scale.

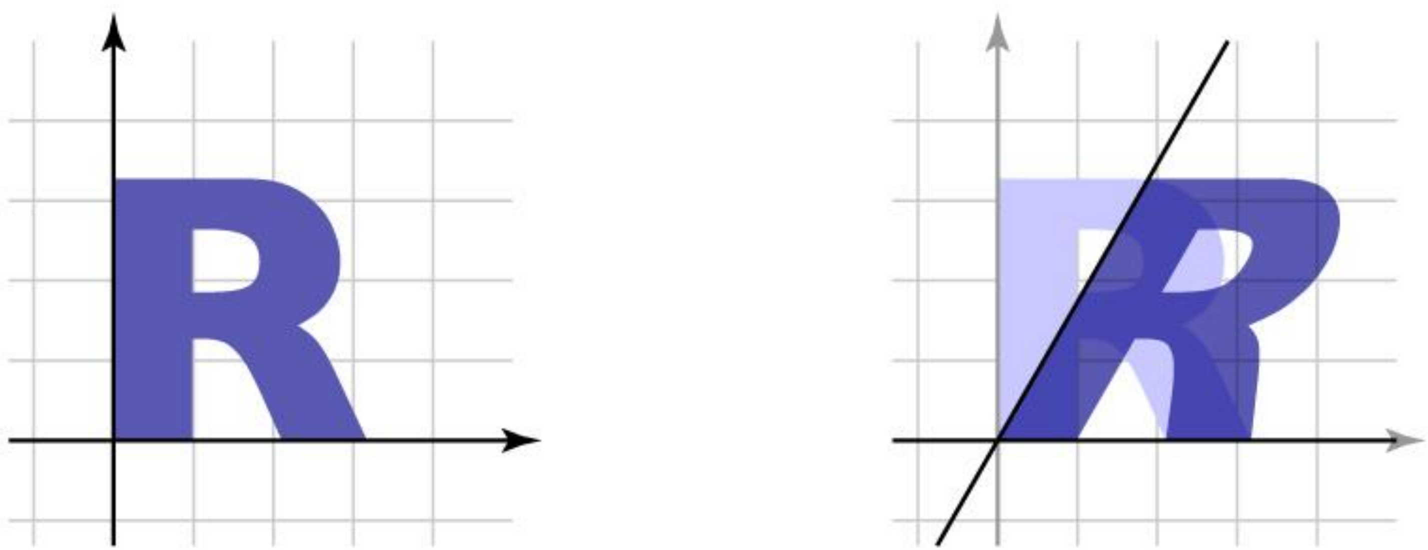


$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D Linear Trans. – Shear

- "Push things sideways"

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$$

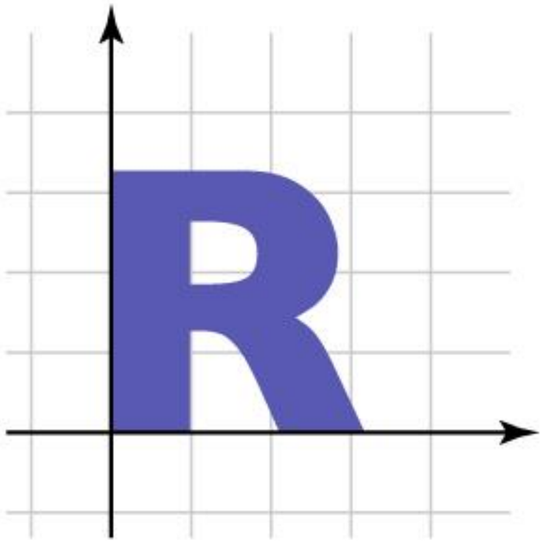


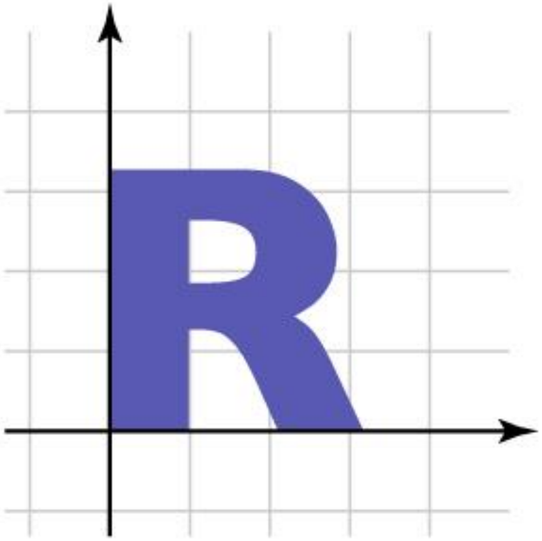
$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

Identity Matrix

- "Doing nothing"

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$





[Practice] Uniform Scale

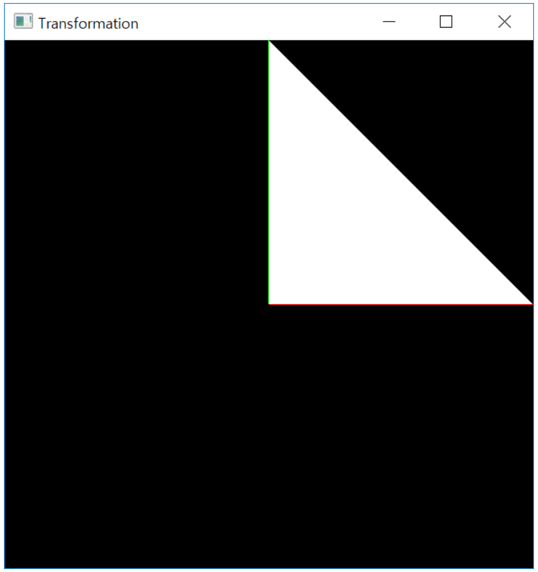
```
import glfw
from OpenGL.GL import *
import numpy as np

def render(M):
    glClear(GL_COLOR_BUFFER_BIT)
    glLoadIdentity()

    # draw cooridnate
    glBegin(GL_LINES)
    glColor3ub(255, 0, 0)
    glVertex2fv(np.array([0.,0.]))
    glVertex2fv(np.array([1.,0.]))
    glColor3ub(0, 255, 0)
    glVertex2fv(np.array([0.,0.]))
    glVertex2fv(np.array([0.,1.]))
    glEnd()

    # draw triangle - p'=Mp
    glBegin(GL_TRIANGLES)
    glColor3ub(255, 255, 255)
    glVertex2fv(M @ np.array([0.0,0.5]))
    glVertex2fv(M @ np.array([0.0,0.0]))
    glVertex2fv(M @ np.array([0.5,0.0]))
    glEnd()
```

[Practice] Uniform Scale




```python
def main():
    if not glfw.init():
        return
    window = glfw.create_window(640,640, "2D Trans", None,None)
    if not window:
        glfw.terminate()
        return
    glfw.make_context_current(window)

    while not glfw.window_should_close(window):
        glfw.poll_events()

        M = np.array([[2.,0.],
                      [0.,2.]])
        render(M)

        glfw.swap_buffers(window)

    glfw.terminate()

if __name__ == "__main__":
    main()
```

[Practice] Animate It!

```
def main():
    if not glfw.init():
        return
    window = glfw.create_window(640,640,"2D Trans", None,None)
    if not window:
        glfw.terminate()
        return
    glfw.make_context_current(window)

    # set the number of screen refresh to wait before calling glfw.swap_buffer().
    # if your monitor refresh rate is 60Hz, the while loop is repeated every 1/60 sec
    glfw.swap_interval(1)

    while not glfw.window_should_close(window):
        glfw.poll_events()

        # get the current time, in seconds
        t = glfw.get_time()

        s = np.sin(t)
        M = np.array([[s,0.],
                      [0.,s]])
        render(M)

        glfw.swap_buffers(window)
    glfw.terminate()
```

[Practice] Nonuniform Scale, Rotation, Reflection, Shear

```
while not glfw.window_should_close(window):
          glfw.poll_events()
          t = glfw.get_time()

          # nonuniform scale
          s = np.sin(t)
          M = np.array([[s,0.],
                        [0.,s*.5]])
          # rotation
          th = t
          M = np.array([[np.cos(th), -np.sin(th)],
                        [np.sin(th), np.cos(th)]])
          # reflection
          M = np.array([[-1.,0.],
                        [0.,1.]])
          # shear
          a = np.sin(t)
          M = np.array([[1.,a],
                        [0.,1.]])
          # identity matrix
          M = np.identity(2)

          render(M)
          glfw.swap_buffers(window)
```

Quiz #1

- Go to https://www.slido.com/
- Join #**cg-ys**
- Click “Polls”

- Submit your answer in the following format:
  - **Student ID: Your answer**
  - **e.g. 2017123456: 4)**

- Note that you must submit all quiz answers in the above format to be checked for “attendance”.

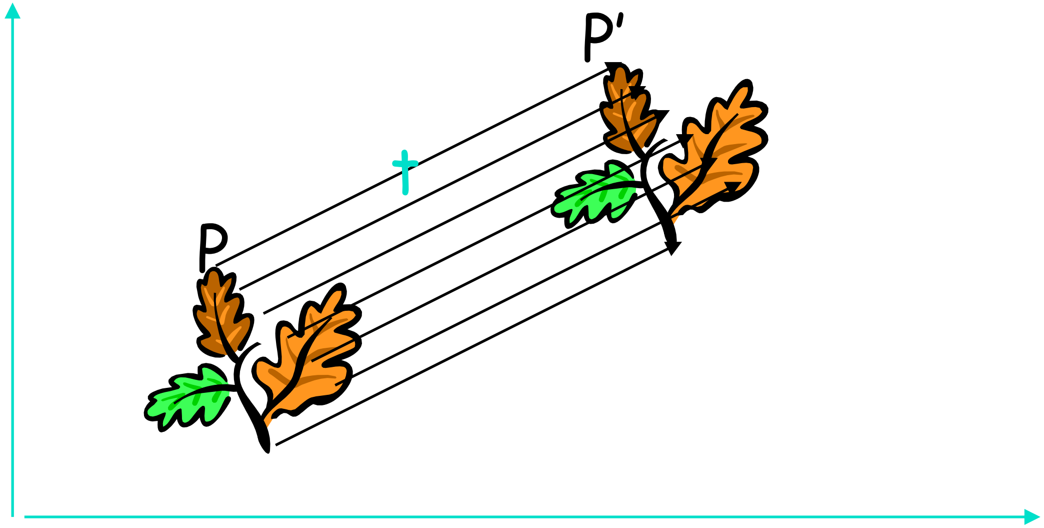
2D Translation

- Translation is the simplest transformation:

  $$T(\mathbf{v}) = \mathbf{v} + \mathbf{u}$$

- Inverse:

  $$T^{-1}(\mathbf{v}) = \mathbf{v} - \mathbf{u}$$



[Practice] Translation

```
def render(u):
    # ...
    glBegin(GL_TRIANGLES)
    glColor3ub(255, 255, 255)
    glVertex2fv(np.array([0.0,0.5]) + u)
    glVertex2fv(np.array([0.0,0.0]) + u)
    glVertex2fv(np.array([0.5,0.0]) + u)
    glEnd()

def main():
    # ...
    while not glfw.window_should_close(window):
        glfw.poll_events()
        t = glfw.get_time()

        u = np.array([np.sin(t), 0.])
        render(u)
        # ...
```

Is translation linear transformation?

- No, because it cannot be represented using a simple matrix multiplication.

- We can express it using vector addition:
$$T(\mathbf{v}) = \mathbf{v} + \mathbf{u}$$

- Combining with linear transformation:
$$T(\mathbf{v}) = M\mathbf{v} + \mathbf{u}$$

➡ **Affine transformation**

Let's check again

- Linear transformation
  - Scale, rotation, reflection, shear
  - Represented as matrix multiplications

$$T(\mathbf{v}) = M\mathbf{v}$$

- Translation
  - Not a linear transformation
  - Can be expressed using vector addition

$$T(\mathbf{v}) = \mathbf{v} + \mathbf{u}$$

Affine Transformation

- Linear transformation + Translation

$$T(\mathbf{v}) = M\mathbf{v} + \mathbf{u}$$

- Preserves lines
- Preserves parallel lines
- Preserves ratios of distance along a line
- → These properties are inherited from linear transformations.

Rigid Transformation

- Rotation + Translation

$$T(\mathbf{v}) = R\mathbf{v} + \mathbf{u}$$, where R is a rotation matrix.

- Preserves distances between all points
- Preserves cross product for all vectors
  - to avoid reflection

[Practice] Affine Transformation

```
def render(M, u):
    # ...
    glBegin(GL_TRIANGLES)
    glColor3ub(255, 255, 255)
    glVertex2fv(M @ np.array([0.0,0.5]) + u)
    glVertex2fv(M @ np.array([0.0,0.0]) + u)
    glVertex2fv(M @ np.array([0.5,0.0]) + u)
    glEnd()

def main():
    # ...
    while not glfw.window_should_close(window):
        glfw.poll_events()
        t = glfw.get_time()

        th = t
        R = np.array([[np.cos(th), -np.sin(th)],
                      [np.sin(th), np.cos(th)]])
        u = np.array([np.sin(t), 0.])
        render(R, u)
        # ...
```

Quiz #2

- Go to https://www.slido.com/
- Join #**cg-ys**
- Click “Polls”

- Submit your answer in the following format:
  - **Student ID: Your answer**
  - **e.g. 2017123456: 4)**

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Composing Transformations & Homogeneous Coordinates

Composing Transformations

- Move an object, then move it some more

$$\mathbf{p} \rightarrow T(\mathbf{p}) \rightarrow S(T(\mathbf{p})) = (S \circ T)(\mathbf{p})$$

- **Composing 2D linear transformations** just works by **2x2 matrix multiplication**

$$T(\mathbf{p}) = M_T\mathbf{p}; S(\mathbf{p}) = M_S\mathbf{p}$$

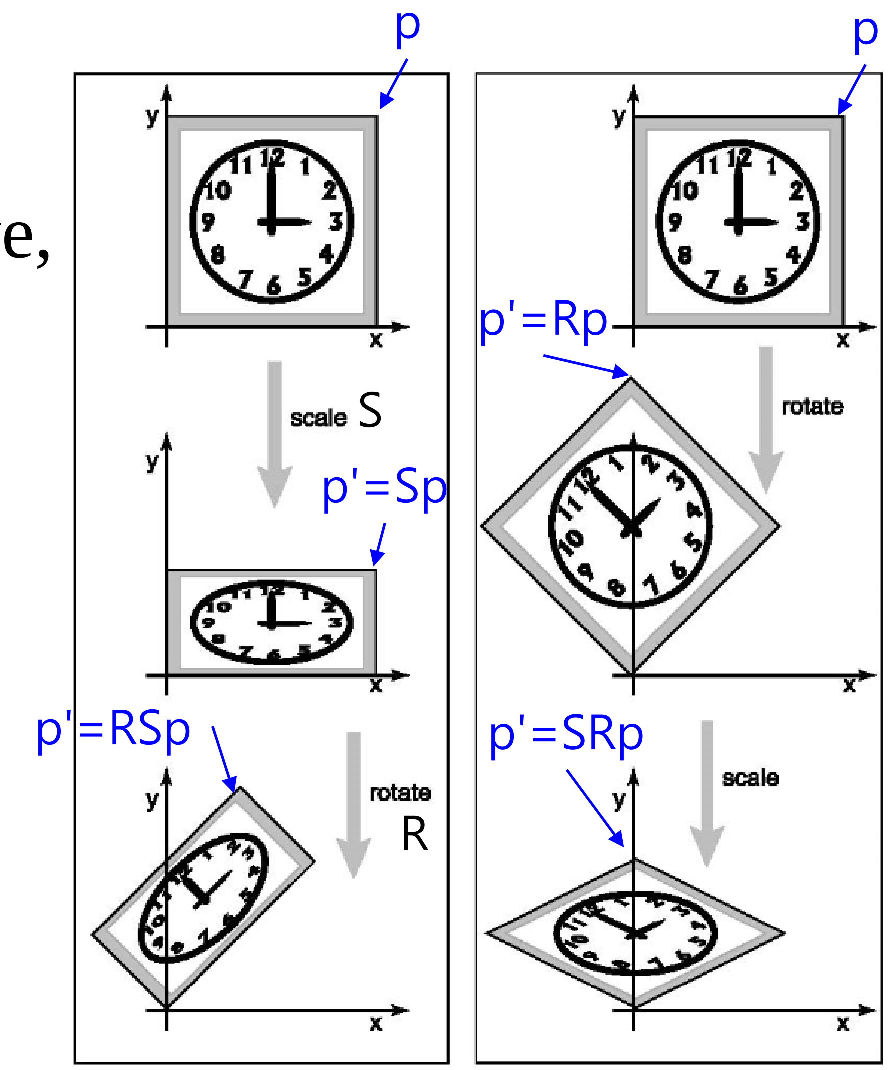
$$(S \circ T)(\mathbf{p}) = M_S M_T \mathbf{p} = (M_S M_T)\mathbf{p} = M_S(M_T\mathbf{p})$$

Order Matters!

- Note that matrix multiplication is associative, but **not commutative**.

$$(AB)C = A(BC)$$
$$AB \neq BA$$

- As a result, the **order of transforms is very important.**



[Practice] Composition

```
def main():
    # ...
    while not glfw.window_should_close(window):
        glfw.poll_events()

        S = np.array([[1.,0.],
                      [0.,2.]])

        th = np.radians(60)
        R = np.array([[np.cos(th), -np.sin(th)],
                      [np.sin(th), np.cos(th)]])

        u = np.zeros(2)

        # compare results of these two lines
        render(R @ S, u)      # p'=RSp
        # render(S @ R, u)      # p'=SRp

        # ...
```

Problems when handling Translation as Vector Addition

- Cannot treat linear transformation (rotation, scale,…) and translation in a consistent manner.

- Composing affine transformations is complicated

$$T(\mathbf{p}) = M_T\mathbf{p} + \mathbf{u}_T \qquad (S \circ T)(\mathbf{p}) = M_S(M_T\mathbf{p} + \mathbf{u}_T) + \mathbf{u}_S$$

$$S(\mathbf{p}) = M_S\mathbf{p} + \mathbf{u}_S \qquad\qquad = (M_S M_T)\mathbf{p} + (M_S\mathbf{u}_T + \mathbf{u}_S)$$

- We need a cleaner way!

  ➡ **Homogeneous coordinates**

Homogeneous Coordinates

- Key idea: Represent 2D points in 3D coordinate space
- Extra component *w* for vectors, extra row/column for matrices
  - For points, can always keep *w* = 1
  - 2D point $[x, y]^T \rightarrow$ 3D vector $[x, y, 1]^T$.
- Linear transformations are represented as:

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- Translations are represented as:

$$\begin{bmatrix} 1 & 0 & \boxed{t} \\ 0 & 1 & \boxed{s} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t \\ y + s \\ 1 \end{bmatrix}$$

- Affine transformations are represented as:

linear part

$$\begin{bmatrix} m_{11} & m_{12} & u_x \\ m_{21} & m_{22} & u_y \\ 0 & 0 & 1 \end{bmatrix}$$

translational part

Homogeneous Coordinates

- **Composing affine transformations** just works by **3x3 matrix multiplication**

$$T(\mathbf{p}) = M_T\mathbf{p} + \mathbf{u}_T$$
$$S(\mathbf{p}) = M_S\mathbf{p} + \mathbf{u}_S$$

$$T(\mathbf{p}) = \begin{array}{c} \begin{array}{cc} \scriptstyle 2\text{x}2 & \scriptstyle 2\text{x}1 \end{array} \\ \begin{bmatrix} M_T & \mathbf{u}_T \\ 0 & 1 \end{bmatrix} \end{array} \qquad S(\mathbf{p}) = \begin{array}{c} \begin{array}{cc} \scriptstyle 2\text{x}2 & \scriptstyle 2\text{x}1 \end{array} \\ \begin{bmatrix} M_S & \mathbf{u}_S \\ 0 & 1 \end{bmatrix} \end{array}$$

Homogeneous Coordinates

- **Composing affine transformations** just works by **3x3 matrix multiplication**

$$
(S \circ T)(\mathbf{p}) = \begin{array}{c} \begin{array}{cc} \text{2x2} & \text{2x1} \end{array} \\ \begin{bmatrix} M_S & \mathbf{u}_S \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{cc} \text{2x2} & \text{2x1} \end{array} \\ \begin{bmatrix} M_T & \mathbf{u}_T \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \text{2x1} \\ \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} \end{array}
$$

$$
= \begin{bmatrix} (M_S M_T)\mathbf{p} + (M_S \mathbf{u}_T + \mathbf{u}_S) \\ 1 \end{bmatrix}
$$

- Much cleaner

[Practice] Homogeneous Coordinates

```
def render(M):
    # ...
    glBegin(GL_TRIANGLES)
    glColor3ub(255, 255, 255)
    glVertex2fv( (M @ np.array([.0,.5,1.]))[:-1] )
    glVertex2fv( (M @ np.array([.0,.0,1.]))[:-1] )
    glVertex2fv( (M @ np.array([.5,.0,1.]))[:-1] )
    glEnd()
```

[Practice] Homogeneous Coordinates

```
def main():
    # ...
    while not glfw.window_should_close(window):
        glfw.poll_events()

        # rotate 60 deg about z axis
        th = np.radians(60)
        R = np.array([[np.cos(th), -np.sin(th),0.],
                      [np.sin(th), np.cos(th),0.],
                      [0.,          0.,         1.]])

        # translate by (.4, .1)
        T = np.array([[1.,0.,.4],
                      [0.,1.,.1],
                      [0.,0.,1.]])

        render(R)    # p'=Rp
        # render(T)   # p'=Tp
        # render(T @ R) # p'=TRp
        # render(R @ T) # p'=RTp
        # ...
```

Summary: Homogeneous Coordinates in 2D

- Use $(\mathbf{x},\mathbf{y},\mathbf{1})^{\mathbf{T}}$ instead of $(x,y)^T$ for **2D points**
- Use **3x3 matrices** instead of 2x2 matrices **for 2D linear transformations**
- Use **3x3 matrices** instead of vector additions **for 2D translations**

- → We can treat linear transformations and translations **in a consistent manner!**

Quiz #3

- Go to https://www.slido.com/
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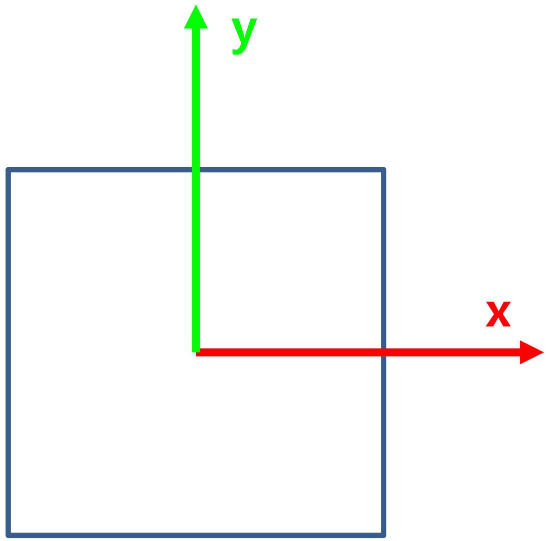
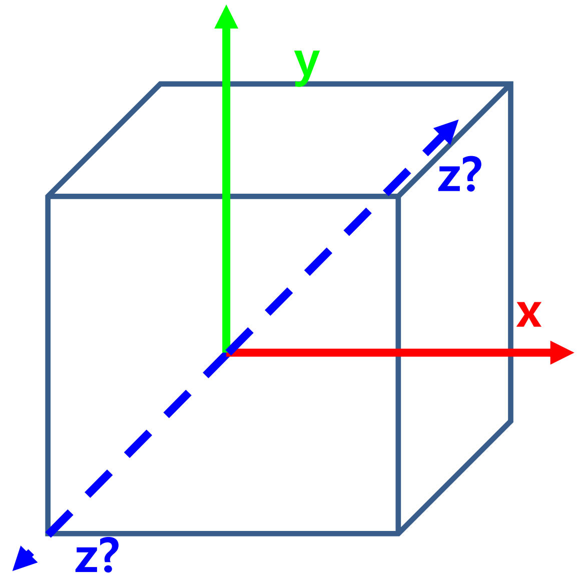
3D Cartesian Coordinate System

Now, Let's go to the 3D world!

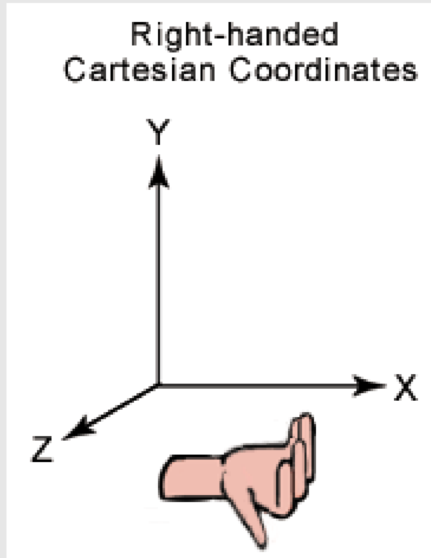
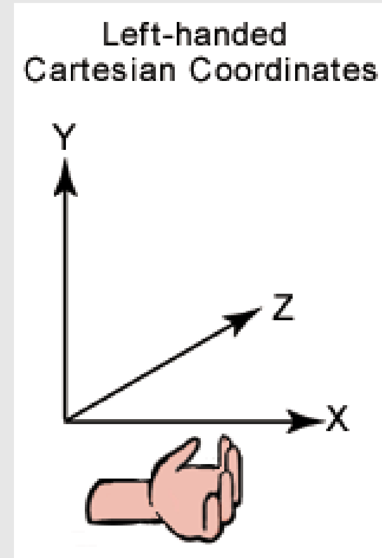




- Coordinate system (좌표계)
  - Cartesian coordinate system (직교좌표계)

Two Types of 3D Cartesian Coordinate Systems

What we're using

		
Positive rotation direction	**counterclockwise** about the axis of rotation 	**clockwise** about the axis of rotation 
Used in…	**OpenGL**, Maya, Houdini, AutoCAD, … Standard for Physics & Math	DirectX, Unity, Unreal, …

Point Representation in Cartesian & Homogeneous Coordinate System

	Cartesian coordinate system	Homogeneous coordinate system
A **2D point** is represented as…	$\begin{bmatrix} p_x \\ p_y \end{bmatrix}$	$\begin{bmatrix} p_x \\ p_y \\ \boxed{1} \end{bmatrix}$
A **3D point** is represented as…	$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$	$\begin{bmatrix} p_x \\ p_y \\ p_z \\ \boxed{1} \end{bmatrix}$

Next Time

- Lab in this week:
  - Lab assignment 3

- Next lecture:
  - 4 - Transformation 2

- Acknowledgement: Some materials come from the lecture slides of
  - Prof. Taesoo Kwon, Hanyang Univ., http://calab.hanyang.ac.kr/cgi-bin/cg.cgi
  - Prof. Steve Marschner, Cornell Univ., http://www.cs.cornell.edu/courses/cs4620/2014fa/index.shtml